Solutions to Homework Set #6
December 3, 2002

Problem 1:

The fields of the TM_{nm0} mode are ($\beta = 0$):

$$E_z = A \sin n\phi \, J_m(k_c \rho)$$

$$H_\rho = \frac{j\omega\epsilon n}{k_c^2 \rho} A \cancel{\cos} n\phi \, J_n(k_c \rho) \qquad k_c = P_{nm}/a = k$$

$$H_\phi = \frac{-j\omega\epsilon}{k_c} A \sin n\phi \, J_m'(k_c \rho)$$

The stored electric energy is;

$$W_e = \frac{\epsilon}{4}\int_V |\bar{E}|^2 dv = \frac{A^2\epsilon}{4}\int_{\rho=0}^{a}\int_{\phi=0}^{2\pi}\int_{z=0}^{d} \sin^2 n\phi \, J_m^2(k_c\rho)\rho \, d\rho \, d\phi \, dz$$

$$= \frac{A^2\epsilon}{4}\pi d \frac{a^2}{2} J_m'(P_{nm}) = \frac{A^2 a^2 \pi d \epsilon}{8} J_m'^2(P_{nm})$$

Note: I used the following integral relations involving Bessel functions:

$$\int^{x} Z_m^2(kx)\, x \, dx = \frac{x^2}{2}\left[Z_n'^2(kx) + \left(1 - \frac{n^2}{k^2x^2}\right) Z_n^2(kx)\right]$$

The power loss due to finite conductivity is,

$$P_\ell = \frac{R_s}{2}\int_S |\bar{H}_t|^2 ds = \frac{R_s}{2}\left\{\int_{\phi=0}^{2\pi}\int_{z=0}^{d} |H_\phi(\rho=a)|^2 a \, d\phi \, dz \right.$$

$$\left. + 2\int_{\rho=0}^{a}\int_{\phi=0}^{2\pi}\left[|H_\rho|^2 + |H_\phi|^2\right]\rho \, d\rho \, d\phi \right\}$$

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$$\Rightarrow P_\ell = \frac{A^2 R_s}{2}\left\{\frac{\pi a d}{\eta^2} J_m'^2(P_{nm}) + \frac{2\pi}{\eta^2}\frac{P_{nm}^2}{2k_c^2} J_m'(P_{nm})\right\}$$

$$= \frac{A^2 R_s \pi}{2\eta^2}(ad + a^2)\, J_m'(P_{nm})$$

Then $Q_c = \dfrac{2\omega W_e}{P_\ell} = \dfrac{\omega a^2 \pi d \epsilon (2\eta^2)}{4 R_s \pi a (d+a)} = \dfrac{adk\eta}{2R_s(d+a)}$

The power ~~lost~~ in the dielectric is,

$$P_d = \frac{\omega\epsilon''}{2}\int_v |\bar{E}|^2 dv = \frac{\omega\epsilon \tan\delta}{2}\int_v |\bar{E}|^2 dv = \frac{2kW_e}{\eta\epsilon}\tan\delta$$

So $Q_d = \dfrac{2\omega W_e}{P_\ell} = \dfrac{1}{\tan\delta}$

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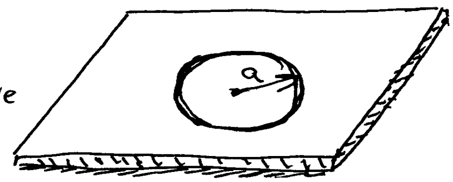
Problem 2:



For TM_{nm0} modes, we have $H_z = 0$ and $\frac{\partial}{\partial z} = 0$.

The wave equation for E_z is:

$$\left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + k^2\right) E_z = 0 \quad (\beta = k)$$

The general solution is,

$$E_z = (A_n \cos n\phi + B_n \sin n\phi)\, J_n(k\rho) \quad (\text{finite at } \rho = 0)$$

Since the choice of $\sin n\phi$ or $\cos n\phi$ (or any combination) depends only on the choice of the $\phi = 0$ reference, we can let $B_n = 0$.

Then $E_z = A_n \cos n\phi\, J_n(k\rho)$

Then (from lecture notes) we can find H_ϕ,

$$H_\phi = -\frac{j\omega\epsilon}{k^2}\frac{\partial E_z}{\partial\rho} = -\frac{j\omega\epsilon}{k} A_n \cos n\phi\, J_n'(k\rho)$$

For $H_\phi = 0$ at $\rho = a$, we require $J_n'(ka) = 0$, or $ka = p'_{nm}$.

So the resonant frequency is

$$f_{nm0} = \frac{ck}{2\pi\sqrt{\epsilon_r}} = \frac{c\, p'_{nm}}{2\pi a\sqrt{\epsilon_r}} \quad \text{and} \quad f_{110} = \frac{c\, p'_{11}}{2\pi a\sqrt{\epsilon_r}} = \frac{1.841\, c}{2\pi a\sqrt{\epsilon_r}}$$

Note: This solution neglects the effect of fringing fields.

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Problem 3:

a) For the WC-109 circular polystyrene-filled waveguide with diameter $2a = 2.779\,cm$, the modes that can propagate at 10 GHz can be TE_{11}, TM_{01}, TE_{21}, TE_{01}, TM_{11}, TE_{31} which have cutoff frequencies given by

$$f_{c\,TE_{11}} \simeq \frac{1.8412(3\times10^{10})}{\pi(2.779)\sqrt{2.56}} \simeq 3.954\ GHz$$

$$f_{c\,TM_{01}} \simeq \frac{2.4049(3\times10^{10})}{\pi(2.779)\sqrt{2.56}} \simeq 5.165\ GHz$$

$$f_{c\,TE_{21}} \simeq \frac{3.0542(3\times10^{10})}{\pi(2.779)\sqrt{2.56}} \simeq 6.559\ GHz$$

$$f_{c\,TE_{01}} \simeq \frac{3.832(3\times10^{10})}{\pi(2.779)\sqrt{2.56}} \simeq 8.23\ GHz$$

$$f_{c\,TE_{31}} \simeq \frac{4.201(3\times10^{10})}{\pi(2.779)\sqrt{2.56}} \simeq 9.022\ GHz$$

Note that the next higher mode (in this case TM_{21}) does not propagate since its cutoff frequency is $\sim 11.03\ GHz > \sim 10\ GHz$

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Problem3: cont.

b) The phase velocity, the guide wavelength, and the wave impedance of the dominant TE_{11} mode at 10 GHz can be calculated as

$$v_{p_{11}} \simeq \frac{(3\times10^{8}\,m s^{-1})/\sqrt{2.56}}{\sqrt{1-\left(\frac{3.954}{10}\right)^2}} \simeq 2.04\times10^{8}\ m/sec$$

$$\lambda_{11} \simeq \frac{(3\,cm)/\sqrt{2.56}}{\sqrt{\left(1-(3.954/10)\right)^2}} \simeq 2.04\ cm$$

$$Z_{TE_{11}} \simeq \frac{(377\,\Omega)/\sqrt{2.56}}{\sqrt{1-(3.954/10)^2}} \simeq 256.5\,\Omega$$

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Problem 4:

a) For the TE_{10p} mode, with p half-wave length in the axial direction, the resonant frequency is

$$\omega_{10p} = \frac{1}{\sqrt{\mu\epsilon}} \sqrt{\frac{\pi^2}{a^2} + \frac{p^2\pi^2}{d^2}}$$

So we have

$$f_{101} \simeq \frac{c}{2\pi} \sqrt{\frac{\pi^2}{(8.636)^2} + \frac{\pi^2}{d^2}} \simeq \frac{3\times10^{10}}{2} \sqrt{\frac{1}{(8.636)^2} + \frac{1}{d^2}} \simeq 2.45\ \text{GHz}$$

from which we find the length of the resonator to be $d \simeq 8.68$ cm

b) Repeating a) for the TE_{102} mode, we have

$$f_{102} \simeq \frac{3\times10^{10}}{2} \sqrt{\frac{1}{(8.636)^2} + \frac{4}{d^2}} \simeq 2.45\ \text{GHz}$$

from which we solve for $d \simeq 17.4$ cm.

c) Repeating a) for TE_{101} mode in a water-filled (assume $\epsilon_r \simeq 10$ at 2.45 GHz) resonant cavity, we have

$$f_{101} \simeq \frac{3\times10^{10}}{2\sqrt{10}} \sqrt{\frac{1}{(8.636)^2} + \frac{1}{d^2}} \simeq 2.45\ \text{GHz}$$

$$\Rightarrow d \simeq 1.99\ \text{cm}$$